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Virtual embeddability between surface mapping class groups

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Coverings of surfaces sometimes induce injective homomorphism from (a finite index subgroup of) a mapping class group into another mapping class group. Using Birman–Hilden double branched coverings and right-angled Artin groups, we can completely determine whether certain mapping class groups is virtually embedded in another mapping class group. Here, a group H is said to be *virtually embedded* in a group G if there is a finite index subgroup of H which is embedded in G.

1 Surface embedding and Birman–Hilden double branched covering

In this article, by $S^b_{g,p}$ we denote a connected orientable surface of genus g with p marked points and with b boundary components. The *homeomorphism group*, $\mathrm{Homeo}_+(S^b_{g,p})$, of the surface $S^b_{g,p}$ is the group of orientation-preserving homeomorphisms of $S^b_{g,p}$, which preserve the set of the marked points and fix the boundary components point-wise. The *mapping class group* $\mathrm{Mod}(S^b_{g,p})$ is the quotient group obtained from $\mathrm{Homeo}_+(S^b_{g,p})$ by collapsing homeomorphisms isotopic to the identity map. In particular, the mapping class group $\mathrm{Mod}(S^1_{0,p})$ is called the *p-th braid group* and is denoted by B_p.

Let us see some typical methods for embedding mapping class groups. The reader is referred to a well-written textbook [5].

1.1 Cylindrical embedding

Let P and P' be the sets of marked points of surfaces S and S', respectively. An inclusion map $\iota\colon S \to S'$ between two surfaces is called a *surface embedding* if $\iota^{-1}(P') \subset P$.

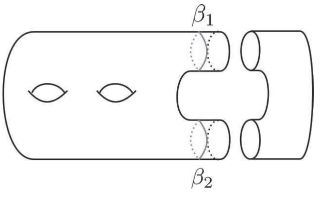

図 1: Essential simple closed curves β_1, β_2 parallel to the boundary components.

We say that a surface embedding ι is *cylindrical* if every component of $S' \setminus \mathrm{Int}N(\iota(S))$ is homeomorphic to none of $S^1_{0,0}$ and $S^1_{0,1}$, and at least one component of $S' \setminus \mathrm{Int}N(\iota(S))$ is a cylinder $S^2_{0,0}$.

Example 1.1. Consider a surface embedding $S^2_{g-1,0} \to S^0_{g,0}$ (see Figure 1) obtained from gluing $S^2_{0,0}$ into $S^2_{g-1,0}$. This surface embedding is cylindrical. Gluing a homeomorphism of $S^2_{g-1,0}$ and the identity map of the cylinder induces a new homeomorphism of $S^0_{g,0}$. Since every homeomorphism isotopic to the identity map of $S^2_{g-1,0}$ induces a homeomorphism isotopic to the identity map of $S^0_{g,0}$, we have a canonical homomorphism $\phi\colon \mathrm{Mod}(S^2_{g-1,0}) \to \mathrm{Mod}(S^0_{g,0})$. The kernel of ϕ is generated by $T_{\beta_1}T_{\beta_2}^{-1}$ (see [13]). Here, T_{β_i} is the Dehn twist along an essential closed curve parallel to a boundary component C_i of $S^2_{g-1,0}$ $(i = 1, 2)$.

1.2 Anannular embedding

A surface embedding $\iota\colon S \to S'$ is said to be *anannular* if each component of $S' \setminus \mathrm{Int}N(\iota(S))$ is homeomorphic to none of $S^1_{0,0}$, $S^1_{0,1}$ and $S^2_{0,0}$.

An anannular surface embedding $S \to S'$ induces [13] an injective homomorphism $\mathrm{Mod}(S) \hookrightarrow \mathrm{Mod}(S')$.

Example 1.2. One-holed sphere with n marked points, $S^1_{0,n}$, admits a surface embedding into a sphere with $n+2$ marked points, $S^0_{0,n+2}$. This surface embedding is anannular, and hence we have $B_n \hookrightarrow \mathrm{Mod}(S^0_{0,n+2})$.

Example 1.3. By gluing $S^2_{0,1}$ into $S^2_{g-1,0}$, we obtain an anannular surface embedding $S^2_{g-1,0} \to S^0_{g,1}$. Hence, $\mathrm{Mod}(S^2_{g-1,0}) \hookrightarrow \mathrm{Mod}(S^0_{g,1})$.

1.3 Birman–Hilden double branched covering

Example 1.4. Assume that $n \leq 2g$.

Take the center line of $S^2_{g-1,0}$ and consider the π-rotation (hyper-elliptic involution) with respect to this center line. Then we have a double branched covering $p\colon S^2_{g-1,0} \to S^1_{0,2g}$.

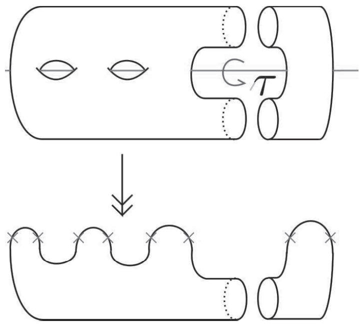

図 2: $S^2_{2,0}$ covers $S^1_{0,6}$ with 6 branched points, and $S^0_{3,0}$ covers $S^0_{0,8}$ with 8 branched points. The symbol τ represents a hyper-elliptic involution, and crossings represent marked points.

By $\mathrm{SHomeo}_+(S^2_{g-1,0})$, we denote the subgroup of $\mathrm{Homeo}_+(S^2_{g-1,0})$, which consists of fiber-preserving homeomorphisms. Here, a homeomorphism $f\colon S^2_{g-1,0} \cong S^2_{g-1,0}$ is said to be *fiber-preserving* if for all $x, x' \in S^2_{g-1,0}$ with $p(x) = p(x')$, the identity $p(f(x)) = p(f(x'))$ holds. Obviously, any fiber-preserving homeomorphism of $S^2_{g-1,0}$ descends to a homeomorphism of $S^1_{0,2g}$. Moreover, every homeomorphism of $S^1_{0,2g}$ has a lift $S^2_{g-1,0} \to S^2_{g-1,0}$, which is a fiber-preserving homeomorphism. Thus, we have a surjective homomorphism $\mathrm{SHomeo}_+(S^2_{g-1,0}) \twoheadrightarrow \mathrm{Homeo}_+(S^1_{0,2g})$. By the Birman–Hilden theory, any fiber-preserving homeomorphism of $S^2_{g-1,0}$, which is isotopic to the identity, is fiber-isotopic to the identity. In other words, if a fiber-preserving homeomorphism of $S^2_{g-1,0}$ is isotopic to the identity, then the descendant in the quotient $S^1_{0,2g}$ must be isotopic to the identity. Hence, we have a canonical surjective homeomorphism

$$d\colon \mathrm{SMod}(S^2_{g-1,0}) \twoheadrightarrow \mathrm{Mod}(S^1_{0,2g}) = B_{2g}.$$

Here, the *symmetric mapping class group* $\mathrm{SMod}(S^2_{g-1,0})$ is the subgroup of $\mathrm{Mod}(S^2_{g-1,0})$ consisting of fiber-preserving mapping classes. This canonical homomorphism d is injective, because every homeomorphism of $S^1_{0,2g}$ has a lift. Thus, we have an embedding

$$d^{-1}\colon B_{2g} \hookrightarrow \mathrm{Mod}(S^2_{g-1,0}).$$

Furthermore, the restriction of ϕ to the symmetric mapping class group $\mathrm{SMod}(S^2_{g-1,0}) \cong B_{2g}$ is injective. To see this, we have to show that $T^m_{\beta_1}T^{-m}_{\beta_2}$ is contained in $\mathrm{SMod}(S^2_{g-1,0})$ only if $m = 1$. We now suppose that $T^m_{\beta_1}T^{-m}_{\beta_2} \in \mathrm{SMod}(S^2_{g-1,0})$. Then $T^m_{\beta_1}T^{-m}_{\beta_2}$ is an element of the center of $\mathrm{SMod}(S_{g-1,0})$. On the other hand, the center of B_{2g} is cyclic and is identified with $\langle T_{\beta_1}T_{\beta_2}\rangle$ in $\mathrm{SMod}(S^2_{g-1,0})$. Now, the assumption $T^m_{\beta_1}T^{-m}_{\beta_2} \in \langle T_{\beta_1}T_{\beta_2}\rangle$ implies $m = 1$. Therefore, the restriction of ϕ is injective and $B_{2g} \hookrightarrow \mathrm{Mod}(S^0_{g,0})$.

Example 1.5. Case $p = 2g+2$. Take the center line of $S^0_{g,0}$ and consider the π-rotation τ with respect to the center line. Then we obtain a double branched covering $p\colon S^0_{g,0} \to S^0_{0,2g+2}$. In this case, τ is an element of $\mathrm{SMod}(S^0_{g,0})$ and descends to the identity map of $S^0_{0,2g+2}$. Hence, by an argument similar as in the previous example, we have an isomorphism

$$\mathrm{SMod}(S^0_{g,0})/\langle\tau\rangle \cong \mathrm{Mod}(S^0_{0,2g+2}).$$

Since $\mathrm{Mod}(S^0_{g,0})$ is residually finite, there is a finite index subgroup H of $\mathrm{Mod}(S^0_{g,0})$ which avoids τ. Then $H \cap \mathrm{SMod}(S^0_{g,0})$ is a finite index subgroup of $\mathrm{SMod}(S^0_{g,0})$ and is embedded in $\mathrm{Mod}(S^0_{0,2g+2})$ as a finite index subgroup. Therefore, $\mathrm{Mod}(S^0_{0,2g+2})$ is virtually embedded in $\mathrm{Mod}(S^0_{g,0})$.

Case $p \le 2g+1$. The pure braid group PB_{p-1} (the kernel of canonical homomorphism from B_{p-1} to the $(p-1)$-th symmetric group Σ_{p-1}) splits as a direct product $PB_{p-1} \cong \mathrm{PMod}(S^0_{0,p}) \times \mathbb{Z}$. By Example 1.4, we have $B_{p-1} \hookrightarrow \mathrm{Mod}(S^0_{g,0})$. Hence, $\mathrm{PMod}(S^0_{0,p})$ is embedded in $\mathrm{Mod}(S^0_{g,0})$. Since $\mathrm{PMod}(S^0_{0,p})$ is a finite index subgroup of $\mathrm{Mod}(S^0_{0,p})$, we have that $\mathrm{Mod}(S^0_{0,p})$ is virtually embedded in $\mathrm{Mod}(S^0_{g,0})$.

The reader is referred to [1], [2] and [7] for unbranched coverings of surface which induce injective homomorphisms between mapping class groups.

Remark 1.6. Suppose that $p \ge 2$. Then the mapping class group $\mathrm{Mod}(S^0_{0,p})$ of a sphere with p marked points has a non-trivial torsion element. On the other hand, B_{p-1} is torsion-free. Hence, $\mathrm{Mod}(S^0_{0,p})$ does not admit an embedding into B_{p-1}. However, a finite index subgroup $\mathrm{PMod}(S^0_{0,p})$ of $\mathrm{Mod}(S^0_{0,p})$ is embedded in B_{p-1}.

Theorem 1.7. *Suppose $g \ge 1$ and $\delta \in \{0,1\}$. Then we have the following.*

(1) If $n \le 2g$, then $B_n \hookrightarrow \mathrm{Mod}(S^0_{g,0})$.

(2) If $p \le 2g+2$, then $\mathrm{Mod}(S^0_{0,p})$ is virtually embedded in $\mathrm{Mod}(S^0_{g,0})$.

For more details about the Birman–Hilden theory, see Margalit–Winarski [12].

2 Right-angled Artin groups

For a simple graph Γ, the *right-angled Artin group* $A(\Gamma)$ on Γ is the group which has the following group presentation:

$$A(\Gamma) = \langle v_1, v_2, \ldots, v_n \mid v_i v_j v_i^{-1} v_j^{-1} = 1 \text{ if } \{v_i, v_j\} \in E(\Gamma)\rangle.$$

Here, $\{v_1, v_2, \ldots, v_n\}$ is the vertex set of Γ and $E(\Gamma)$ is the edge set of Γ.

One algebraic virtue of right-angled Artin group is as follows.

Lemma 2.1. *Let A be right-angled Artin group, G a group, and H a finite index subgroup of G. If A is embedded in G, then A is embedded in H.*

Theorem 2.2 ([11, Theorem 1.1])**.** *For a sufficiently large n, the n-th powers of the Dehn twists $T_1^n, \ldots, T_m^n$ along mutually non-isotopic essential simple closed curves generate a right-angled Artin group in* $\mathrm{Mod}(S_{g,p}^0)$

3 Obstructions to the existence of virtual embeddings

The *cohomological dimension* $\mathrm{cd}(G)$ of a group G is defined to be the maximum dimension n such that the n-th group cohomology $H^n(G, M)$ is non-trivial for some G-module M. By Serre's theorem [14], the cohomological dimension of a torsion-free group G coincides with that of any finite index subgroup of G. Note that the mapping class group of surfaces have torsion-free subgroups of finite indices. Hence, the *virtual cohomological dimension* $\mathrm{vcd}(\mathrm{Mod}(S))$ of the mapping class group $\mathrm{Mod}(S)$, which is defined to be the cohomological dimension of a torsion-free finite index subgroup of $\mathrm{Mod}(S)$, is well-defined. The virtual cohomological dimensions of the mapping class groups are computed by Harer.

Theorem 3.1 ([6, Theorem 4.1])**.** *Suppose that $2g + p + b > 2$. Then we have*

$$\mathrm{vcd}(\mathrm{Mod}(S_{g,p}^b)) = \begin{cases} 4g - 5 & (p + b = 0) \\ 4g + p + 2b - 4 & (p + b > 0) \\ p + 2b - 3 & (g = 0) \end{cases}$$

If a group H is virtually embedded in a group G, then $\mathrm{vcd}(H) \leq \mathrm{vcd}(G)$.

Example 3.2. B_{p+1} is not embedded in $\mathrm{Mod}(S_{0,p+2}^0)$ even virtually, because $\mathrm{vcd}(B_{p+1}) = p > p - 1 = \mathrm{vcd}(\mathrm{Mod}(S_{0,p+2}^0))$.

Theorem 3.3. *$B_n \hookrightarrow \mathrm{Mod}(S_{0,p+2}^0)$ if and only if $n \leq p + 2$.*

Proof. Suppose that $n \leq p + 2$. Then $S_{0,n}^1$ admits an anannular embedding into $S_{0,p+2}^0$. Hence, $B_n \hookrightarrow \mathrm{Mod}(S_{0,p+2}^0)$.

We now suppose that $B_n \hookrightarrow \mathrm{Mod}(S_{0,p+2}^0)$. Then by Example 3.2, we have $n \leq p+2$. □

Theorem 3.4 ([4, Theorem A])**.** *Suppose that $2 - 2g - p < 0$. If G is an abelian subgroup of $\mathrm{Mod}(S_{g,p}^0)$, then G is finitely generated with torsion-free rank bounded by $3g - 3 + p$.*

4 Certain right-angled Artin groups in mapping class groups

In this section we introduce embeddability results on certain right-angled Artin groups. By P_m we denote the path graph on m vertices (the underlying space of P_m is homeomorphic to the unit closed interval). By C_m we denote the cyclic graph on m vertices (the underlying space of C_m is homeomorphic to the unit circle). *Complement graph* Γ^c of the original graph Γ is the graph whose vertex set is $V(\Gamma)$ and the edge set is $\{\{u,v\} \mid \{u,v\} \notin E(\Gamma)\}$.

Theorem 4.1. $A(P_m^c) \hookrightarrow \mathrm{Mod}(S_{g,p}^0)$ *if and only if* m *satisfies the following inequality.*

$$m \leq \begin{cases} 0 & ((g,p) \in \{(0,0),(0,1),(0,2),(0,3)\}) \\ 2 & ((g,p) \in \{(0,4),(1,0),(1,1)\}) \\ p-1 & (g=0,\ p \geq 5) \\ p+2 & (g=1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2). \end{cases}$$

Theorem 4.2. $A(C_m^c) \hookrightarrow \mathrm{Mod}(S_{g,p}^0)$ *if and only if* m *satisfies*

$$m \leq \begin{cases} 3 & (g,p) = (1,1) \\ 5 & (g,p) = (1,2) \\ p+2 & (g=1,\ p \geq 3) \\ 2g+p+1 & (g \geq 2,\ 1 \leq p \leq 2). \\ 2g+p & (g \geq 2,\ p \geq 3). \end{cases}$$

Theorem 4.3. *Suppose that* $p \geq 2$. *Then the following hold.*

(1) $A(P_m^c) \hookrightarrow B_p$ *if and only if* m *satisfies*

$$m \leq \begin{cases} p-1 & (p = 2,3) \\ p & (p \geq 4). \end{cases}$$

(2) $A(C_m^c) \hookrightarrow B_p$ *if and only if* m *satisfies*

$$m \leq \begin{cases} 0 & (p = 2) \\ 3 & (p = 3) \\ p+1 & (p \geq 4). \end{cases}$$

Theorem 4.4. $A(C_m^c) \times \mathbb{Z} \hookrightarrow B_p$ *if and only if* m *satisfies*

$$m \leq \begin{cases} 0 & (p = 2) \\ 3 & (p = 3) \\ p+1 & (p \geq 4). \end{cases}$$

Theorem 4.5. *Let g be an integer ≥ 2. Then $A(C_m^c) \times \mathbb{Z} \hookrightarrow \mathrm{Mod}(S_{g,p}^0)$ if and only if m satisfies*

$$m \leq \begin{cases} 2g+1 & (g \geq 2,\ p=0) \\ 2g+p & (g \geq 2,\ p \geq 1). \end{cases}$$

The reader is referred to the papers [9] and [10] for more details. The "if parts" of Theorems come from Koberda's embedding theorem (Theorem 2.2) together with desired curve systems on surfaces. The "only if parts" are derived from the combinatorial structure of curve graphs of surfaces.

5 Main Theorem

Theorem 5.1. *Suppose that $g \geq 1$ and $\delta \in \{0,1\}$.*

(1) B_n is virtually embedded in $\mathrm{Mod}(S_{g,\delta}^0)$ if and only if $n \leq 2g$.

(2) $\mathrm{Mod}(S_{0,p}^0)$ is virtually embedded in $\mathrm{Mod}(S_{g,0}^0)$ if and only if $p \leq 2g+2$.

Proof. (1) The "if part" follows from Theorem 1.7 (1). Therefore, we will prove the "only if" part. Suppose that B_n is virtually embedded in $\mathrm{Mod}(S_{g,\delta}^0)$. Namely, there is a finite index subgroup H of B_n which is embedded in $\mathrm{Mod}(S_{g,\delta}^0)$

Case $g=1$. B_3 contains free abelian group $\mathbb{Z}^2$ of rank two as a subgroup. However, $\mathrm{Mod}(S_{1,0}^0) = \mathrm{Mod}(S_{1,1}^0) = \mathrm{SL}(2,\mathbb{Z})$ does not contain $\mathbb{Z}^2$ as a subgroup. Hence, B_3 is not virtually embedded in $\mathrm{Mod}(S_{1,0}^0) = \mathrm{Mod}(S_{1,1}^0)$. Thus, $n \leq 2$, as required.

Case $g \geq 2$. Assuming $n \geq 4$, we will prove that $n \leq 2g$. Since $n \geq 4$, the braid group B_n contains $A(C_{n+1}) \times \mathbb{Z}$. Hence, H contains $A(C_{n+1}) \times \mathbb{Z}$ as a subgroup by Lemma 2.1. By our assumption that H is embedded in $\mathrm{Mod}(S_{g,\delta}^0)$. Thus we have $n+1 \leq 2g+1$ (*i.e.*, $n \leq 2g$).

(2) can be treated similarly. □

Problems concerning virtual embeddability between mapping class groups can be found in [3] and [8].

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